A General Framework for Solving Singular SPDEs

Feng-Yu Wang

(Tianjin University)

17th Workshop on Markov Processes and Related Topics

A joint work with Hao Tang (University of Oslo)

arXiv: 2208.08312

Outline

- ♣ Motivation: SPDE with pseudo-differential noise
- ♣ A general frame work for singular SPDEs
- ♣ SPDEs with pseudo-differential noise
- ♣ Application to specific models

1. Introduction: transport noise

• Flandoli/Gubinelli/Priola [Inv. Math. 10]: stochastic transport equation for $X : [0,T] \times \Omega \to L^\infty(\mathbb{R}^d)$:

$$\mathrm{d}X(t) = \big\{b(t,\cdot)\cdot\nabla X(t)\big\}\mathrm{d}t + \sum_{i=1}^d \{\partial_i X(t)\} \circ \mathrm{d}W^i(t),$$

$b \in L^1_{loc}(([0,T]\times\mathbb{R}^d;\mathbb{R}^d)$ with $\nabla\cdot b \in L^1_{loc}([0,T]\times\mathbb{R}^d)$, $\{W^i(t)\}$ are independent 1D Brownian motions.

A number of nonlinear SPDEs with transport (i.e. first order differential) noise have been intensively investigated:

- Burgers: Alonson-Orán/de León/Takao [NoDEA'19]
- 2D Euler: Flandoli/Luo [AOP'20], Lang/Crisan [SPDE-AC'22]; 3D Euler: Crisan/Flandoli/Holm [JNS'19].
- 3D Navier-Stokes: Flandoli/Luo [PTRF'21],
- Hunter-Saxton: Holden/Karlsen/Pang [JDE'21].
- General equation in Hilbert space:
  Flandoli/Galeati/Luo [CPDE'21];
  Alonso-Orán/Rohde/Tang [JNS'21]...

1. Introduction: motivation

The transport noise is given by first order derivatives (i.e. vector fields), and hence has 1-order singularity;
it maps from $H^s(=W^{s,2})$ to H^{s-1} for any $s \geq 0$.
We intend to consider noise with arbitrary order singularity.

Let $\mathbb{K}^d = \mathbb{R}^d$ or $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$, and let $s > 0, d, m \in \mathbb{N}$.
We aim to study SPDEs on $\mathbb{H} := \Pi H^s(\mathbb{K}^d; \mathbb{R}^m)$ driven by

$$\sum_{k=1}^{\infty} \{\Pi \mathcal{A}_k X(t)\} \circ \mathrm{d}W_k(t) + \tilde{h}(t, X(t))\mathrm{d}\tilde{W}(t),$$

singular noise + regular noise

- Π is a projection (Leray projection for NS/Euler equations, zero-average operator for functions on $\mathbb{T}^d$, identity operator);
- $\{\mathcal{A}_k\}$ are pseudo-dimensional operators;
- $\{W_k(t)\}$ are independent 1-D Brownian motions, $\tilde{W}(t)$ is a cylindrical Brownian motion on $\mathbb{H}$ independent of $\{W_k(t)\}$;
- $\tilde{h}(t, X_t)$ takes value as Hilbert-Schmidt operators in the state space $\mathbb{H}$.

A general framework of proper regularization

Let $\mathbb{H} \hookrightarrow \mathbb{M}$ be two separable Hilbert spaces with continuous and dense embedding, let

$$\mathcal{W}(t) := \sum_{k \geq 1} W_k(t)\, e_k$$

be cylindrical Brownian motion on a separable Hilbert space $\mathbb{U}$ with ONB $\{e_k\}_{k \geq 1}$, where $\{W_k(t)\}$ are independent 1-D Brownian motions. Consider the following equation on $\mathbb{H}$:

$$(E1) \qquad \mathrm{d}X(t) = g(t, X(t))\mathrm{d}t + h(t, X(t))\mathrm{d}\mathcal{W}(t),$$

- $g : [0, T] \times \mathbb{H} \to \mathbb{M}$, (singular)
- $h : [0, T] \times \mathbb{H} \to \mathcal{L}_2(\mathbb{U}; \mathbb{M})$. (singular)

$\mathcal{L}_2(\mathbb{U}; \mathbb{M})$: the space of Hilbert-Schmidt operators form $\mathbb{U}$ to $\mathbb{M}$.

Solution

We call (X,τ) a maximal solution to $(E1)$, if

1. τ a stopping time with $\mathbb{P}(\tau>0)=1$,
2. $[0,\tau)\ni t\mapsto X(t)\in\mathbb{H}$ is adapted and weakly continuous,
3. $\mathbb{P}$-a.s.

$$\sup_{s\in[0,t]}\|X(s)\|_{\mathbb{H}}<\infty,\ \ t\in[0,\tau),$$

$$\limsup_{t\uparrow\tau}\|X(t)\|_{\mathbb{H}}=\infty \text{ on } \{\tau<\infty\},$$

and the following equation holds on $\mathbb{M}$ (NOT $\mathbb{H}$):

$$X(t)=X(0)+\int_0^t g(s,X(s))\mathrm{d}s + \int_0^t h(s,X(s))\mathrm{d}\mathcal{W}(s),\quad t\in[0,\tau).$$

The solution is called non-explosive, if $\mathbb{P}(\tau=\infty)=1$.

Proper regularization

$\{(g_n, h_n)\}_{n\geq 1}$ is called a proper regularization of (g, h), if

$$g_n : [0,\infty)\times\mathbb{H}\to\mathbb{H},\quad h_n : [0,\infty)\times\mathbb{H}\to\mathcal{L}_2(\mathbb{U};\mathbb{H}),\quad n\geq 1$$

are measurable such that the following conditions hold for some increasing $K : [0,\infty)\times[0,\infty)\to[0,\infty)$ and a dense $\mathbb{M}_0\subset\mathbb{M}$:

(1) For any $t\geq 0$ and $X\in\mathbb{H}$,

$$\sup_{n\geq 1}\Big\{\|g_n(t,X)\|_{\mathbb{M}}+\|h_n(t,X)\|_{\mathcal{L}_2(\mathbb{U};\mathbb{M})}\Big\}\leq K(t,\|X\|_{\mathbb{M}})(1+\|X\|_{\mathbb{H}}),$$

$$\lim_{n\to\infty}\big\{\|g_n(t,X)-g(t,X)\|_{\mathbb{M}}+\|h_n(t,X)-h(t,X)\|_{\mathcal{L}_2(\mathbb{U};\mathbb{M})}\big\}=0.$$

(2) For any $n, N\geq 1$,

$$\sup_{X\neq Y;\, t,\|X\|_{\mathbb{H}},\|Y\|_{\mathbb{H}}\leq N}\Bigg\{\|g_n(t,0)\|_{\mathbb{H}}+\|h_n(t,0)\|_{\mathcal{L}_2(\mathbb{U};\mathbb{H})}+\frac{\|g_n(t,X)-g_n(t,Y)\|_{\mathbb{H}}+\|h_n(t,X)-h_n(t,Y)\|_{\mathcal{L}_2(\mathbb{U};\mathbb{H})}}{\|X-Y\|_{\mathbb{H}}}\Bigg\}<\infty.$$

Proper regularization

(3) For any $Y \in \mathbb{M}_0$, $T > 0$ and $\{X_n, X\}_{n\ge1} \subset \mathcal{B}_b([0,T];\mathbb{H}) \cap C([0,T];\mathbb{M})$ with $X_n \to X$ in $C([0,T];\mathbb{M})$ as $n \to \infty$,

$$\lim_{n\to\infty} \int_0^T \Big\{ \Big| \big\langle g_n(t, X_n(t)) - g(t, X(t)), Y \big\rangle_{\mathbb{M}} \Big| + \sum_{k\ge1} \big\langle \big\{ h_n(t, X_n(t)) - h(t, X(t)) \big\} e_k, Y \big\rangle_{\mathbb{M}}^2 \Big\} \mathrm{d}t = 0.$$

(4) (cancellation of singularities) For any $t \ge 0$ and $X \in \mathbb{H}$,

$$\sup_{n\ge1} \sum_{k=1}^{\infty} \langle h_n(t,X) e_k, X \rangle_{\mathbb{H}}{}^2 \le K(t, \|X\|_{\mathbb{M}})(1 + \|X\|_{\mathbb{H}}{}^4),$$

$$\sup_{n\ge1} \Big\{ 2 \langle g_n(t,X), X \rangle_{\mathbb{H}} + \|h_n(t,X)\|_{\mathcal{L}_2(\mathbb{U};\mathbb{H})}^2 \Big\} \le K(t, \|X\|_{\mathbb{M}})(1 + \|X\|_{\mathbb{H}}{}^2), \quad Y \in \mathbb{M}_0.$$

Assumption: local well-posedness

(A) (g, h) has a proper regularization $\{(g_n, h_n)\}_{n\geq 1}$ satisfying the (asymptotic quasi monotonicity) condition:

There exist increasing function $K : [0,\infty) \times [0,\infty) \to (0,\infty)$, *and a function* $\lambda : \mathbb{N} \times \mathbb{N} \to [0,\infty)$ *with* $\lim_{n,l\to\infty} \lambda_{n,l} = 0$, *such that for any* $X \neq Y \in \mathbb{H}$,

$$\max\Big\{ \sum_{k\geq 1} \frac{\langle \{h_n(t,X) - h_l(t,X)\} e_k, X - Y\rangle_{\mathbb{M}}^2}{\|X - Y\|_{\mathbb{M}}^2},$$

$$2\langle g_n(t,X) - g_l(t,Y), X - Y\rangle_{\mathbb{M}} + \|h_n(t,X) - h_l(t,Y)\|_{\mathcal{L}_2(\mathbb{U};\mathbb{M})}^2 \Big\}$$

$$\leq K(t, \|X\|_{\mathbb{H}} + \|Y\|_{\mathbb{H}}) \big(\lambda_{n,l} + \|X - Y\|_{\mathbb{M}}^2\big), \quad n, l \geq 1,\ t \geq 0.$$

Note:

1. For $\lambda_{j,l} = 0$ and constant K, the condition becomes monotonicity in $\mathbb{M}$.
2. Since $\|\cdot\|_{\mathbb{M}} \lesssim \|\cdot\|_{\mathbb{H}}$, even for $\lambda_{n,l} = 0$, this condition is weaker than local monotonicity in $\mathbb{M}$.

Assumption: strong continuity

This assumption implies the existence and uniqueness of maximal solution. For the continuity in $\mathbb{H}$, we assume

(B) There exists $\{T_n\}_{n\geq 1} \subset \mathcal{L}(\mathbb{M};\mathbb{H})$ (space of bounded linear operators form $\mathbb{M}$ to $\mathbb{H}$), such that

$$\lim_{n\to\infty} \|T_n X - X\|_{\mathbb{H}} = 0, \quad X \in \mathbb{H},$$

and for all $t \geq 0,\ N \geq 1$,

$$\sup_{n\geq 1, \|X\|_{\mathbb{H}}\leq N} \Big\{ 2\,\langle T_n g(t,X), T_n X\rangle_{\mathbb{H}} + \|T_n h(t,X)\|^2_{\mathcal{L}_2(\mathbb{U};\mathbb{H})}, \sum_{i=1}^{\infty} \langle T_n h(t,X)e_i, T_n X\rangle^2_{\mathbb{H}} \Big\} \leq K(t,N).$$

Note: This condition implies the continuity of $t \mapsto \|X(t)\|_{\mathbb{H}}$, which together with the weak continuity of $X(t)$ in $\mathbb{H}$, implies the strong continuity.

Assumption: global well-posedness

Finally, to prove the non-explosion, we assume the following Lyapunov type condition. When $V'' < 0$, a fast enough growth of the noise coefficient will kill the growth of other terms, such that the non-explosion is ensured.

(C) There exists a function $1 \le V \in C^2([0,\infty))$ satisfying

$$V'(r) > 0, \quad V''(r) \le 0, \quad \lim_{r\to\infty} V(r) = \infty,$$

such that for some function $0 \le F \in L^1_{loc}([0,\infty))$ and for all $(t,X) \in [0,\infty) \times \mathbb{H}$,

$$V'(\|X\|_{\mathbb{M}}^2)\Big\{2\big\langle b(t,X)+g(t,X),X\big\rangle_{\mathbb{M}} + \|h(t,X)\|^2_{\mathcal{L}_2(\mathbb{U};\mathbb{M})}\Big\}$$

$$+\,2V''(\|X\|_{\mathbb{M}}^2)\sum_{k=1}^{\infty}\langle h(t,X)e_k,X\rangle_{\mathbb{M}}^2 \le F(t)V(\|X\|_{\mathbb{M}}^2).$$

Note: This condition comes from Itô's formula for $V(\|X(t)\|_{\mathbb{M}}^2)$ of the solution. Although the solution takes values in $\mathbb{H}$, the proper regularization allows to prove non-explosion using the smaller norm $\|\cdot\|_{\mathbb{M}}$.

Main result

Theorem (1)

Assume **(A)** *and let* $X(0)$ *be an* $\mathcal{F}_0$*-measurable* $\mathbb{H}$*-valued random variable.*

(1) $(E1)$ *has a unique maximal solution* (X, τ), *and the solution satisfies* $\mathbb{P}$*-a.s.*

$$\limsup_{t\uparrow\tau} \|X(t)\|_{\mathbb{M}} = \infty \ \ on\ \{\tau < \infty\}.$$

(*stronger than definition:* $\|\cdot\|_{\mathbb{M}} \lesssim \|\cdot\|_{\mathbb{H}}$)

(2) *If* **(B)** *holds, then the solution is continuous in* $\mathbb{H}$.

(3) *If* **(C)** *holds, then the solution is non-explosive.*

3. SPDE with pseudo-differential noise

We first introduce pseudo-differential operators on $\mathbb{R}^d$. Those on $\mathbb{T}^d$ can be defined correspondingly but with $\xi \in \mathbb{Z}^d$ rather than $\mathbb{R}^d$ for the Fourier transform. Denote

$$|\alpha|_1 := \sum_{k=1}^d \alpha_k, \quad \partial^\alpha := \prod_{k=1}^d \partial_k^{\alpha_k}, \quad \alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{Z}_+^d.$$

When different variables $x, \xi \in \mathbb{R}^d$ appear, we use ∂_x^α and ∂_ξ^α to denote ∂^α in x and ξ respectively. For any $s \in \mathbb{R}$, we define two classes of s-order symbols

$$S^s := \Big\{ \wp \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{C}) \ : \ |\wp|_{\alpha,\beta;s} := \sup_{x,\xi\in\mathbb{R}^d} \frac{|\partial_x^\beta \partial_\xi^\alpha \wp(x,\xi)|}{(1+|\xi|)^{s-|\alpha|_1}} < \infty, \ \ \alpha, \beta \in \mathbb{Z}_+^d \Big\},$$

$$S_0^s := \big\{ \wp \in S^s(\mathbb{R}^d \times \mathbb{R}^d) : \ \wp(x,\xi) = \wp(\xi) \big\}.$$

A set $\mathcal{D} \subset S^s$ is called bounded, if

$$\sup_{\wp\in\mathcal{D}} |\wp|_{\alpha,\beta;s} < \infty, \ \ \alpha, \beta \in \mathbb{Z}_+^d.$$

3. SPDE with pseudo-differential noise

For any $\wp \in S^s$, the pseudo-differential operator $\mathrm{OP}(\wp)$ with symbol $\wp$ is defined as

$$[\mathrm{OP}(\wp)f](x) := \int_{\mathbb{R}^d} \wp(x,\xi)\widehat{f}(\xi)\mathrm{e}^{2\pi\mathrm{i}(x\cdot\xi)}\mathrm{d}\xi, \quad x \in \mathbb{R}^d,$$
$$\widehat{f}(\xi) := (2\pi)^d \int_{\mathbb{R}^d} f(x)\mathrm{e}^{-2\pi\mathrm{i}(x\cdot\xi)}\mathrm{d}x, \quad \mathrm{i} := \sqrt{-1}.$$

In the following we only consider real-valued operators, i.e., $[\mathrm{OP}(\wp)f]$ is real if so is f; equivalently,

$$(C) \qquad \wp(x,-\xi) = \overline{\wp(x,\xi)} := \mathrm{Re}[\wp(x,\xi)] - \mathrm{Im}[\wp(x,\xi)]\,\mathrm{i}, \;\; x,\xi \in \mathbb{K}^d.$$

Let

$$\mathrm{OP}S^s := \big\{\mathrm{OP}(\wp) \;:\; \wp \in S^s \text{ satisfying } (C)\big\},$$
$$\mathrm{OP}S_0^s := \big\{\mathrm{OP}(\wp) \;:\; \wp \in S_0^s \text{ satisfying } (C)\big\}.$$

A set of pseudo-differential operators are called bounded, if so is the set of their symbols.

3. SPDE with pseudo-differential noise

Recall that $\mathbb{K} = \mathbb{R}$ or $\mathbb{T}$. Let $d, m \in \mathbb{N}$ and $s > 0$. We will consider SPDEs on the Hilbert space

$$\mathbb{H} := \Pi H^s(\mathbb{K}^d; \mathbb{R}^m),$$

for a symmetric projection operator Π satisfying certain conditions. Typical examples of Π include

1. $\Pi = I$: the identity operator;
2. $\Pi = \Pi_L$: the Leray projection for $m = d$:

$$\Pi_L H^s(\mathbb{K}^d; \mathbb{R}^d) = H^s_{\mathrm{div}}(\mathbb{K}^d; \mathbb{R}^d)$$
$$:= \big\{X \in H^s(\mathbb{K}^d; \mathbb{R}^d) : \ \nabla \cdot X = 0\big\}, \ s \geq 0,$$

   where $\nabla \cdot X := \sum_{k=1}^d \partial_k X_k$ is the divergence of $X = (X_k)_{1 \leq k \leq d}$ defined in the weak sense;
3. $\Pi = \Pi_0$: the zero-average projection on $\mathbb{T}^d$:

$$\Pi_0 f := f - \int_{\mathbb{T}^d} f(x) \mathrm{d}x.$$

3. SPDE with pseudo-differential noise

Consider the following stochastic equation on $\mathbb{H}$:

$$(E2) \qquad \mathrm{d}X(t) = g(X(t))\mathrm{d}t + \sum_{k=1}^{\infty} \Big[\{\Pi \mathcal{A}_k X(t)\} \circ \mathrm{d}W_k(t) + \tilde{h}_k(t, X(t))\mathrm{d}\tilde{W}_k(t) \Big],$$

$\{W_k, \tilde{W}_k\}_{k\geq 1}$ are a family of independent 1-D Brownian motions,

$$\tilde{h}_k(t,\cdot) : \mathbb{H} \to \mathbb{H}, \quad k \geq 1, \; t \geq 0$$

are locally Lipschitz continuous, (b, g) comes from deterministic nonlinear PDEs, and

$$\mathcal{A}_k X := \big(a_k \mathcal{Q}_{k,j} X_j + \tilde{a}_k \tilde{\mathcal{Q}}_{k,j} X_j\big)_{1\leq j\leq m},$$
$$k \geq 1, \; X = (X_j)_{1\leq j\leq m} \in \mathbb{H}$$

for some constants $\{a_k, \tilde{a}_k\} \subset \mathbb{R}$, pseudo-differential operators $\{\mathcal{Q}_{k,j}\} \subset \mathrm{OP}S_0^r$ and $\{\tilde{\mathcal{Q}}_{k,j}\} \subset \mathrm{OP}S^{r_0}$ for some $r \geq r_0 \in [0,1]$.

Assumption on pseudo-differential noise

(D) Let $\mathcal{A}^*$ be the L^2-adjoint of a densely defined linear operator $\mathcal{A}$ in L^2.

(1) $\{a_k, \tilde{a}_k\} \subset \mathbb{R}$ are constants such that $a_k \tilde{a}_k = 0$ and

$$\sum_{k \geq 1} \left(a_k^2 + \tilde{a}_k^2\right) < \infty.$$

(2) $\{\mathcal{Q}_{k,j}\} \subset \mathrm{OP}S_0^r$ is bounded, $\{\tilde{\mathcal{Q}}_{k,j}\} \subset \mathrm{OP}S^{r_0}$ is bounded, and there exist a class of bounded zero-order operators $\{\mathcal{T}_{k,j}, \tilde{\mathcal{T}}_{k,j}\}_{k,j} \subset \mathrm{OP}S^0$ such that

$$\mathcal{Q}_{k,j}^* = \mathcal{T}_{k,j} - \mathcal{Q}_{k,j}, \quad \tilde{\mathcal{Q}}_{k,j}^* = \tilde{\mathcal{T}}_{k,j} - \tilde{\mathcal{Q}}_{k,j}.$$

Note:

1. $a_k \tilde{a}_k = 0$ avoids possible higher order operators $[\mathcal{Q}_{k,i}, \tilde{\mathcal{Q}}_{k,j}] := \mathcal{Q}_{k,i}\tilde{\mathcal{Q}}_{k,j} - \tilde{\mathcal{Q}}_{k,j}\mathcal{Q}_{k,i}$ in the quadratic variation.
2. As extensions to the anti-symmetric operators $\{\partial_i\}$ in the transport noise, $\{\mathcal{Q}_{k,j}, \tilde{\mathcal{Q}}_{k,j}\}$ are anti-symmetric up to zero-order operators.

Assumption on regular noise

(E) Let $\mathbb{H} = \Pi H^s(\mathbb{K}^d; \mathbb{R}^m)$ and $\mathbb{M} = \Pi H^\kappa(\mathbb{K}^d; \mathbb{R}^m)$ for some $s > 2r$ and $\kappa \in [0, s-2r)$, where $r \geq r_0$ is in **(D)**.

There exists an increasing function

$$K : [0,\infty) \times [0,\infty) \to [0,\infty)$$

such that for any $t \geq 0,\ X, Y \in \mathbb{H}$,

$$\sum_{k\geq 1} \|\tilde{h}_k(t,X)\|_{\mathbb{H}}^2 \leq K(t, \|X\|_{\mathbb{M}})(1 + \|X\|_{\mathbb{H}}^2),$$

$$\sum_{k=1}^{\infty} \|\tilde{h}_k(t,X) - \tilde{h}_k(t,Y)\|_{\mathbb{H}}^2 \leq K(t, \|X\|_{\mathbb{H}} + \|Y\|_{\mathbb{H}}) \|X - Y\|_{\mathbb{H}}^2,$$

$$\sum_{k=1}^{\infty} \|\tilde{h}_k(t,X) - \tilde{h}_k(t,Y)\|_{\mathbb{M}}^2 \leq K(t, \|X\|_{\mathbb{H}} + \|Y\|_{\mathbb{H}}) \|X - Y\|_{\mathbb{M}}^2.$$

Theorem (2)

Assume **(D)** *and* **(E)**. *If for* $J_n := e^{\frac{1}{n}\Delta}$,

$$g_n(X) := J_n g(J_n X), \quad n \geq 1$$

is a proper regularization of g, *then:*

(1) *For any* $\mathcal{F}_0$*-measurable* $X(0)$ *in* $\mathbb{H}$, *the equation* $(E2)$ *has a unique maximal solution, and* $\limsup_{t\uparrow\tau} \|X(t)\|_{\mathbb{M}} = \infty$ *on* $\{\tau < \infty\}$.

(2) *The solution is continuous in* $\mathbb{H}$, *if there exists an increasing function* $\tilde{K} : [0,\infty) \to [0,\infty)$ *such that*

$$\sup_{n\geq 1} |\langle J_n g(X), J_n X\rangle_{\mathbb{H}}| \leq \tilde{K}(\|X\|_{\mathbb{H}}), \quad X \in \mathbb{H}.$$

It is non-explosive, if there exists $0 \leq F \in L^1_{loc}([0,\infty))$ *such that*

$$\frac{2\langle b(t,X) + g(X), X\rangle_{H^\kappa} + \sum_k \Big(\|\tilde{h}_k(t,X)\|^2_{H^\kappa} - \frac{2\langle \tilde{h}_k(t,X), X\rangle^2_{H^\kappa}}{e + \|X\|^2_{H^\kappa}}\Big)}{(e + \|X\|^2_{H^\kappa}) \log(e + \|X\|^2_{H^\kappa})}$$
$$\leq F(t), \quad t \geq 0.$$

Application to concrete models: MHD

From now on, we assume that the noise coefficients satisfy **(D)** and **(E)** for $\mathbb{H}, \mathbb{M}$ given in each specific model.

Magnetohydrodynamics equation (MHD) : let

$$d \geq 2, \ m = 2d, \ \alpha_1, \ \alpha_2 \in [0,1], \ \mu_1, \mu_2 \geq 0,$$

Let $\Pi = (\Pi_L, \Pi_L)$, where Π_L is the Leray projection, such that

$$\mathbb{H} := \Pi H^s(\mathbb{K}^d; \mathbb{R}^{2d}) = H^s_{\rm div}(\mathbb{R}^d; \mathbb{R}^d) \times H^s_{\rm div}(\mathbb{R}^d; \mathbb{R}^d),$$

and for $X := (X_1, X_2) \in \mathbb{H}$, we define

$$\mathbf{A}(X) := \big(\mu_1(-\Delta)^{\alpha_1} X_1, \ \mu_2(-\Delta)^{\alpha_2} X_2\big),$$
$$\mathbf{B}(X) := \Pi\big((X_2 \cdot \nabla) X_2 - (X_1 \cdot \nabla) X_1, \ (X_2 \cdot \nabla) X_1 - (X_1 \cdot \nabla) X_2\big).$$

The **(MHD)** equation reads

$$\frac{\mathrm{d}}{\mathrm{d}t} X(t) = g^{\rm mhd}(X(t)) := \mathbf{B}(X(t)) - \mathbf{A} X(t).$$

Application to concrete models: MHD

When $X_2 \equiv 0$ and $\alpha_1 = 1$, this equation reduces to the Navier-Stokes ($\mu_1 > 0$) equation

$$\frac{\mathrm{d}}{\mathrm{d}t} X_1(t) = \mu_1 \Delta X_1(t) - \Pi_L \{X_1(t) \cdot \nabla\} X_1(t),$$

and the Euler ($\mu_1 = 0$) equation

$$\frac{\mathrm{d}}{\mathrm{d}t} X_1(t) = -\Pi_L \{X_1(t) \cdot \nabla\} X_1(t), \quad t \geq 0.$$

Let

$$\alpha_0 := 1 + \max_{i=1,2} (2\alpha_i - 1)^+ 1_{\{\mu_i > 0\}}.$$

Application to concrete models: MHD

Theorem (MHD)

For any fixed $s > 1 + \frac{d}{2} + [\alpha_0 \vee (2r)],\ 1 + \frac{d}{2} < \kappa < s - [\alpha_0 \vee (2r)]$, *let*

$$\mathbb{H} := H^s_{\rm div}(\mathbb{K}^d; \mathbb{R}^d) \times H^s_{\rm div}(\mathbb{K}^d; \mathbb{R}^d),$$
$$\mathbb{M} := H^\kappa_{\rm div}(\mathbb{K}^d; \mathbb{R}^d) \times H^\kappa_{\rm div}(\mathbb{K}^d; \mathbb{R}^d),$$
$$g = g^{mhd}.$$

Then for any $\mathcal{F}_0$*-measurable* $X(0)$ *in* $\mathbb{H}$, $(E2)$ *has a unique maximal solution, which is continuous in* $\mathbb{H}$ *and*

$$\lim_{t \uparrow \tau} \|X(t)\|_{\mathbb{M}} = \infty \ \ on\ \{\tau < \infty\}.$$

It is non-explosive if locally uniformly in $t \geq 0$,

$$\lim_{\|X\|_{\mathbb{M}} \to \infty} \frac{1}{\|X\|^3_{\mathbb{M}}} \sum_{k \geq 1} \left(\|\tilde{h}_k(t, X)\|^2_{\mathbb{M}} - \frac{2\langle \tilde{h}_k(t, X), X \rangle^2_{\mathbb{M}}}{1 + \|X\|^2_{\mathbb{M}^2}} \right) < -1.$$

Application to concrete models: KdV/Burgers

Korteweg-De Vries (KdV): $d = m = 1$, $\mathbb{K} = \mathbb{R}$, $\Pi = \mathbf{I}$ and

$$g(X) = g^{\mathrm{kdv}}(X) := -X\partial X - \partial^3 X,$$

where $\partial := \frac{\mathrm{d}}{\mathrm{d}x}$ on $\mathbb{R}$.

Theorem (KdV)

Let $\mathbb{H} = H^s(\mathbb{R};\mathbb{R}), \mathbb{M} = H^\kappa(\mathbb{R};\mathbb{R})$ *for any fixed* s, κ *such that*

$$s > \frac{3}{2} + [3 \vee (2r)], \quad \frac{3}{2} < \kappa < s - [3 \vee (2r)].$$

Then all assertions in Theorem (MHD) *hold for SPDE* $(E2)$ *with*

$$g = g^{\mathrm{kdv}}.$$

Burgers: $g^{bg}(X) = -X\partial X + \nu\partial^2 X, \quad \nu \geq 0.$
The assertions hold for $2 \vee (2r)$ replacing $3 \vee (2r)$.

Application to concrete models: CH

Camassa-Holms (CH): $d = m = 1$, $\mathbb{K} = \mathbb{R}$, $\Pi = \mathbf{I}$ and

$$g(X) = g^{\mathrm{ch}}(X) := -X\partial X - \partial(I-\Delta)^{-1}\Big(\sum_{i=1}^{4} c_i X^i + c|\partial X|^2\Big),$$

where $\{c, c_i\}_{1\le i\le 4}$ are constants.

Theorem (CH)

Let $\mathbb{H} = H^s(\mathbb{R};\mathbb{R}), \mathbb{M} = H^\kappa(\mathbb{R};\mathbb{R})$ *for some*

$$s > \frac{3}{2} + [1 \vee (2r)], \quad \frac{3}{2} < \kappa < s - [1 \vee (2r)].$$

Then all assertions in Theorem (MHD) *hold for SPDE* $(E2)$, *except that the non-explosion condition is strengthened as*

$$\lim_{\|X\|_{\mathbb{M}}\to\infty} \frac{1}{\|X\|_{\mathbb{M}}^5} \sum_{k\ge 1} \bigg(\|\tilde h_k(t,X)\|_{\mathbb{M}}^2 - \frac{2\langle \tilde h_k(t,X), X\rangle_{\mathbb{M}}^2}{1+\|X\|_{\mathbb{M}^2}^2} \bigg) < -1.$$

Application to concrete models: AD

Aggregation-diffusion (AD): $d \geq 2, m = 1, \Pi = \mathbf{I}$.
Let $\mathcal{K} \in \mathcal{S}(\mathbb{K}^d)$ (rapidly decreasing interaction kernel) such that

$$\mathcal{B} := \nabla\{\mathscr{K} *\} \in \mathrm{OP}S^{-1},$$

where $\{\mathscr{K} *\} f(x) := \int_{\mathbb{K}^d} \mathcal{K}(x-y) f(y) \mathrm{d}y$.

The Aggregation-diffusion equation reads

$$\frac{\mathrm{d}}{\mathrm{d}t} X(t) = g^{\mathrm{ad}}(X(t)),$$
$$g^{\mathrm{ad}}(X) := -\nu(-\Delta)^{\beta} - \gamma \nabla \cdot (X \mathcal{B} X),$$

where $\beta \in [0,1], \nu \geq 0$ and $0 \neq \gamma \in \mathbb{R}$ are constants. Let

$$\beta_0 := 1 + (2\beta - 1)^+ 1_{\{\nu > 0\}}.$$

Application to concrete models: **AD**

Theorem (AD)

Let $\mathbb{H} = H^s(\mathbb{K}^d; \mathbb{R}), \mathbb{M} = H^\kappa(\mathbb{K}^d; \mathbb{R})$ *for some*

$$s > 1 + \frac{d}{2} + [\beta_0 \vee (2r)], \quad 1 + \frac{d}{2} < \kappa < s - [\beta_0 \vee (2r)].$$

Then all assertions in Theorem (MHD) *hold for SPDE* (*E*2) *with*

$$g = g^{\mathrm{ad}}.$$

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Application to concrete models: SQG

Surface quasi-geostrophic (SQG): $d = 2, m = 1$ and

$$\Pi = \begin{cases} \mathbf{I}, & \text{if } \mathbb{K}^2 = \mathbb{R}^2, \\ \Pi_0, & \text{if } \mathbb{K}^2 = \mathbb{T}^2. \end{cases}$$

Let $\mathcal{R}$ be the Riesz transform on ΠH^0:

$$\mathcal{R}X := \big(-\partial_2(-\Delta)^{-\frac{1}{2}}X,\ \partial_1(-\Delta)^{-\frac{1}{2}}X\big).$$

the SQG equation reads

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = g^{\mathrm{sqg}}(X(t)) := \Pi\big\{(\mathcal{R}X(t)) \cdot \nabla X(t)\big\}.$$

Application to concrete models: **SQG**

Theorem (SQG)

Let $\mathbb{H} = \Pi H^s(\mathbb{K}^2; \mathbb{R}), \mathbb{M} = \Pi H^\kappa(\mathbb{K}^2; \mathbb{R})$ *for some*

$$s > 1 + \frac{d}{2} + [\beta_0 \vee (2r)], \quad 1 + \frac{d}{2} < \kappa < s - [\beta_0 \vee (2r)].$$

Then all assertions in Theorem (MHD) *hold for SPDE* (*E*2) *with*

$$g = g^{\mathrm{ad}}.$$

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Thank You